

# Influence of Rashba and Dresselhaus spin-orbit interactions of equal strengths on electron states in a circular quantum ring in the presence of a magnetic field

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Solutions of the Schrödinger equation are obtained for an electron in a two-dimensional circular semiconductor quantum ring in the presence of both external uniform constant magnetic field and the Rashba and Dresselhaus spin-orbit interactions of equal strengths. Confinement is simulated by a realistic potential well of a finite depth. The dependence of the energy levels on a magnetic field strength, strength of spin-orbit interaction and a relative ring width is presented.

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## I. INTRODUCTION

At present it is well established that motion of an electron in an inner layer of a semiconductor heterostructure can be treated as a two-dimensional in  $(x, y)$  plane because of the existence of the confining quantum well along  $z$  axis directed perpendicular to  $(x, y)$  plane [1, 2]. In connection with the development of nanotechnology, the study of quantum dots and rings in heterostructures acquires increasing importance. In [3, 4], a simple but fairly adequate model was proposed in which a two-dimensional circular quantum ring corresponds to an axially symmetric rectangular potential well

$$V_c(\rho) = \begin{cases} V, & 0 < \rho < \rho_i, \\ 0, & \rho_i < \rho < \rho_o, \\ V, & \rho_o < \rho < \infty \end{cases} \quad (1)$$

of a finite depth  $V$ , where  $\rho = \sqrt{x^2 + y^2}$ ,  $\rho_i$  and  $\rho_o$  are inner and outer radii of a ring. Notice, that the confining potential of a finite depth was also used for the description of the quantum dots [5, 6].

The influence of the spin-orbit Rashba [7, 8] and Dresselhaus [9] interactions on the electron states in planar heterostructures are widely studied in recent years. A uniform constant magnetic field  $B$  normal to the plane of the quantum ring is described by the vector potential  $\mathbf{A} = \frac{B}{2}(-y, x, 0)$ . Then the interactions of Rashba  $V_R$  and Dresselhaus  $V_D$  are represented by the formulas

$$V_R = \alpha_R(\sigma_x P_y - \sigma_y P_x)/\hbar, \quad V_D = \alpha_D(\sigma_x P_x - \sigma_y P_y)/\hbar, \quad (2)$$

where  $\mathbf{P} = \mathbf{p} + q_e \mathbf{A}$ ,  $q_e$  is the absolute value of the electron charge,  $\sigma_x$  and  $\sigma_y$  are standard Pauli spin-matrices. The strengths of these interactions depend on the used materials. The contribution of two spin-orbit interactions can be measured with applying various experimental methods [2, 10].

In general, the full spin-orbit interaction has the form  $V_R + V_D$  with  $\alpha_R \neq \alpha_D$ . However, the considerable attention is spared to the special case [2, 11, 12], when the spin-orbit interactions of Rashba and Dresselhaus have equal strength  $\alpha_R = \alpha_D = \alpha$ , which can be experimentally achieved due to the fact that the Rashba interaction strength can be controlled by external electric field and the Dresselhaus interaction strength can be varied by changing the width of the quantum well along the  $z$  axis [1, 2]. In the present paper for the special case  $\alpha_R = \alpha_D$ , we obtain the wave functions as well as the dependence of the energy levels on a magnetic field, a spin-orbit interaction strength and a relative width of the quantum ring which is described by the potential (1). The calculations are performed for the parameter values associated with GaAs.

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## II. EXACT SOLUTION OF THE UNPERTURBED SCHRODINGER EQUATION WITHOUT THE ZEEMAN INTERACTION

The total Hamiltonian of the problem can be written as a sum

$$H = H_0 + H' \quad (3)$$

of the unperturbed Hamiltonian

$$H_0 = \frac{P_x^2 + P_y^2}{2M_{\text{eff}}} + V_c(\rho) + \frac{\alpha}{\hbar}(\sigma_x - \sigma_y)(P_x + P_y), \quad (4)$$

where  $M_{\text{eff}}$  is the effective electron mass, and the perturbation describing the Zeeman interaction

$$H' = \frac{1}{2}g\mu_B B\sigma_z, \quad (5)$$

where  $g$  is the effective gyromagnetic factor,  $\mu_B = \frac{q_e\hbar}{2M_e}$  is the Bohr magneton,  $M_e$  is the electron mass,  $\sigma_z$  is the Pauli matrix.

We shall solve the full Schrödinger equation

$$H\Psi = E\Psi \quad (6)$$

in two stages. First, we shall obtain an exact solution of the unperturbed Schrodinger equation

$$H_0\Psi_0 = E_0\Psi_0, \quad (7)$$

and then we shall take into account the Zeeman interaction within the framework of the perturbation theory.

It is easy to see that in the case of the unperturbed equation (7) with the Hamiltonian (4), in addition to the obvious integral of motion

$$\sigma = \frac{\sigma_x - \sigma_y}{\sqrt{2}} \quad (8)$$

there is also a non-trivial integral of motion

$$L = L_z + \alpha M_{\text{eff}}(x - y)(\sigma_x - \sigma_y)/\hbar, \quad (9)$$

where  $L_z$  is the operator of angular momentum.

We look for the solutions of Eq. (7) which are eigenfunctions of operators  $\sigma$  and  $L$ . Then the required solutions admit a factorization of the form

$$\Psi_0^\pm(x, y) = \mathbf{n}^\pm \exp \left[ \mp i\sqrt{2}\alpha M_{\text{eff}}(x + y)/\hbar^2 \right] e^{im\phi} u(\rho), \quad m = 0, \pm 1, \pm 2, \dots, \quad (10)$$

where  $m$  is the angular momentum quantum number,  $\mathbf{n}^\pm$  are eigenvectors of the operator  $\sigma$ :

$$\sigma \mathbf{n}^\pm = \pm \mathbf{n}^\pm, \quad \mathbf{n}^\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm e^{-i\pi/4} \end{pmatrix}. \quad (11)$$

Here we use the polar coordinates  $\rho, \phi$  ( $x = \rho \cos \phi, y = \rho \sin \phi$ ).

Introducing dimensionless quantities

$$r = \frac{\rho}{\rho_o}, \quad e_o = \frac{2M_{\text{eff}}\rho_o^2}{\hbar^2} E_o, \quad v = \frac{2M_{\text{eff}}\rho_o^2}{\hbar^2} V, \quad a = \frac{2M_{\text{eff}}\rho_o}{\hbar^2} \alpha, \quad b = \frac{q_e\rho_o^2}{2\hbar} B, \quad (12)$$

we write the radial equation

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \left( e_0 + a^2 - v_c(r) - \frac{m^2}{r^2} - 2bm - b^2 r^2 \right) u = 0, \quad (13)$$

where

$$v_c(r) = \begin{cases} v, & 0 < r < r_i, \\ 0, & r_i < r < 1, \\ v, & 1 < r < \infty. \end{cases} \quad (14)$$

Here we use the notation  $r_i = \rho_i/\rho_o$  for the relative width of the ring.

In the case of a rectangular potential well (14) in each of the three regions ( $0 < r < r_i$ ,  $r_i < r < 1$ ,  $1 < r < \infty$ ), Eq. (13) coincides with the equation for the radial wave function of an electron in a uniform magnetic field without taking into account the spin-orbit interaction [13], provided we replace  $e_0$  to  $e_0 + a^2 - v_c(r)$ . Therefore, following [13], we represent a required function  $u(r)$  by the formula

$$u(r) = (br^2)^{\frac{|m|}{2}} \exp\left(\frac{-br^2}{2}\right) w(r), \quad (15)$$

where

$$w(r) = \begin{cases} c_1 w_1(r), & 0 < r < r_i, \\ c_{21} w_{21}(r) + c_{22} w_{22}(r), & r_i < r < 1, \\ c_3 w_3(r), & 1 < r < \infty. \end{cases} \quad (16)$$

Here,  $c_1$ ,  $c_{21}$ ,  $c_{22}$  and  $c_3$  are arbitrary coefficients, and the functions  $w_1(r)$ ,  $w_{21}(r)$ ,  $w_{22}(r)$  and  $w_3(r)$  are expressed in terms of the confluent hypergeometric functions of the first and second kind  $M(\alpha, \beta, \xi)$  and  $U(\alpha, \beta, \xi)$  [14] as follows

$$\begin{aligned} w_1(r) &= M(\gamma_o, \beta, br^2), \\ w_{21}(r) &= M(\gamma_i, \beta, br^2), \quad w_{22}(r) = U(\gamma_i, \beta, br^2), \\ w_3(r) &= U(\gamma_o, \beta, br^2), \end{aligned} \quad (17)$$

where

$$\begin{aligned} \gamma_o &= \frac{m + |m| + 1}{2} - \frac{e_0 + a^2 - v}{4b}, \\ \gamma_i &= \frac{m + |m| + 1}{2} - \frac{e_0 + a^2}{4b}, \\ \beta &= |m| + 1. \end{aligned} \quad (18)$$

The particular solutions in the first and third regions are chosen so that the radial wave function is regular at the origin  $r \rightarrow 0$  and tends to zero at the infinity  $r \rightarrow \infty$ .

The continuity conditions for the radial wave function  $u(r)$  and its first derivative  $u'(r) = du(r)/dr$  at the boundary points  $r = r_i$  and  $r = 1$  lead to a system of algebraic equations

$$M_4(m, e_0, v, a, b, r_i) \mathbf{X} = 0 \quad (19)$$

for four coefficients, where  $\mathbf{X} = \{c_1, c_{21}, c_{22}, c_3\}$  and  $M_4(m, e_0, v, a, b, r_i)$  is  $4 \times 4$  matrix of the form

$$M_4 = \begin{pmatrix} w_1(r_i) & -w_{21}(r_i) & -w_{22}(r_i) & 0 \\ w_1'(r_i) & -w_{21}'(r_i) & -w_{22}'(r_i) & 0 \\ 0 & w_{21}(1) & w_{22}(1) & -w_3(1) \\ 0 & w_{21}'(1) & w_{22}'(1) & -w_3'(1) \end{pmatrix}. \quad (20)$$

Therefore, the exact equation for determining  $e_0(m, v, a, b, r_i)$  reads

$$\det M_4(m, e_0, v, a, b, r_i) = 0. \quad (21)$$

From Eq. (13), we see that the dependence of  $e_0$  on  $a$  is trivial  $e_0(m, v, a, b, r_i) = e_0(m, v, 0, b, r_i) - a^2$ . In addition, the following relation  $e_0(m, v, a, b, r_i) - e_0(-m, v, a, b, r_i) = 4bm$  is fulfilled. Of course, Eq. (21) cannot be solved analytically, but can be easily solved numerically.

In order to construct the radial wave function completely, we find the values of required coefficients

$$\begin{pmatrix} c_{21} \\ c_{22} \\ c_3 \end{pmatrix} = -c_1 M_3^{-1}(m, e_0, v, a, b, r_i) \begin{pmatrix} w'_1(r_i) \\ 0 \\ 0 \end{pmatrix}, \quad (22)$$

where

$$M_3 = \begin{pmatrix} -w'_{21}(r_i) & -w'_{22}(r_i) & 0 \\ w_{21}(1) & w_{22}(1) & -w_3(1) \\ w'_{21}(1) & w'_{22}(1) & -w'_3(1) \end{pmatrix}. \quad (23)$$

The residual arbitrariness in the choice of the coefficient  $c_1$  is used to implement the standard normalization condition  $\langle \Psi_0^\pm | \Psi_0^\pm \rangle = 1$ .

### III. CONTRIBUTION OF THE ZEEMAN INTERACTION WITHIN THE FRAMEWORK OF THE PERTURBATION THEORY

The expression for the Zeeman interaction in dimensionless quantities takes the form

$$h' = \frac{2M_{\text{eff}}\rho_o^2}{\hbar^2} H' = 4sb\sigma_z, \quad s = \frac{gM_{\text{eff}}}{4M_e}. \quad (24)$$

Since each energy level of the unperturbed system is doubly degenerate with two eigenfunctions (10), we consider the contribution of Zeeman interaction with the help of the perturbation theory in the degenerate case.

Because of  $\sigma_z \mathbf{n}^\pm = \mathbf{n}^\mp$  in the basis of the eigenvectors  $|\Psi_0^+\rangle$  and  $|\Psi_0^-\rangle$  of the unperturbed Hamiltonian, we have the following equalities

$$\langle \Psi_0^\pm | \sigma_z | \Psi_0^\pm \rangle = 0. \quad (25)$$

for the diagonal matrix elements. Off-diagonal matrix elements are given by

$$\langle \Psi_0^+ | \sigma_z | \Psi_0^- \rangle = \langle \Psi_0^- | \sigma_z | \Psi_0^+ \rangle = \delta(m, v, a, b, r_i), \quad (26)$$

where

$$\delta(m, v, a, b, r_i) = \frac{\int_0^\infty J_0(2ar)u^2(r)r dr}{\int_0^\infty u^2(r)r dr} \quad (27)$$

is expressed in terms of the Bessel function.

Then we get splitting

$$e^\pm = e_0 \pm e' \quad (28)$$

for the unperturbed energy levels, where

$$e' \equiv e'(m, v, a, b, r_i) = 4sb\delta(m, v, a, b, r_i). \quad (29)$$

The following relation  $e'(-m, v, a, b, r_i) = e'(m, v, a, b, r_i)$  is fulfilled for the corrections  $e'$ . Normalized eigenfunctions in zero-order approximation, which correspond to the eigenvalues  $e^\pm$ , are described by the formulas

$$\Psi^\pm = \frac{1}{\sqrt{2}} (\Psi_0^+ \pm \Psi_0^-). \quad (30)$$

Note that in the limiting case  $\alpha = 0$  the expressions (28) and (30) become exact.

#### IV. NUMERICAL RESULTS

Now we present the results of the calculations of the quantities  $e_0$  and  $e'$ . In accordance with [15] we choose the parameter values  $M_{\text{eff}} = 0.067M_e$ ,  $g = -0.44$  related to GaAs. Then we get  $s = -0.00737$ . If we assume  $\rho_o = 30$  nm, then the following correspondences  $a = 1 \rightarrow \alpha = 18.9579$  meV nm,  $e = 1 \rightarrow E = 0.631933$  meV between the dimensionless and dimensional quantities are obtained. For example, at chosen parameters, the dimensionless value  $v = 400$  corresponds to the potential well depth  $V = 252.772$  meV, which is close to the value 257 meV in [4].

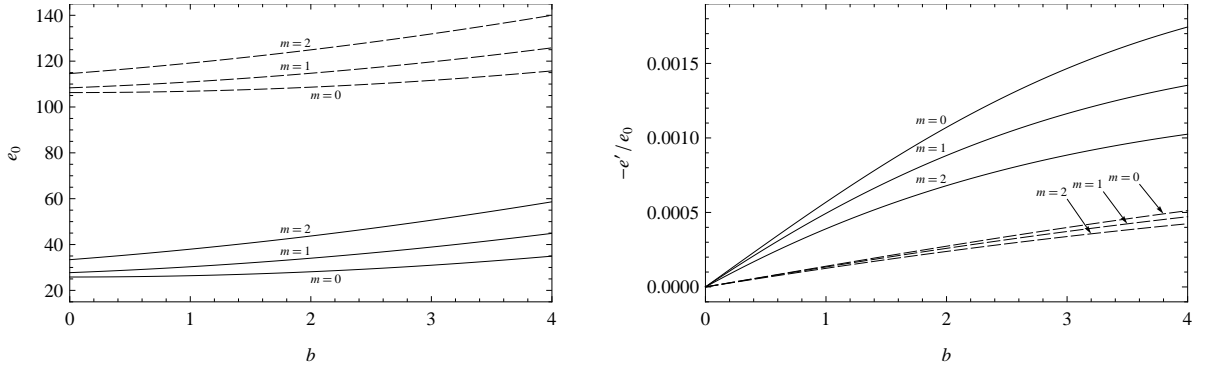


FIG. 1: Dependence of  $e_0$  and  $-e'/e_0$  on  $b$  at  $a = 1$  and  $r_i = 0.5$ .

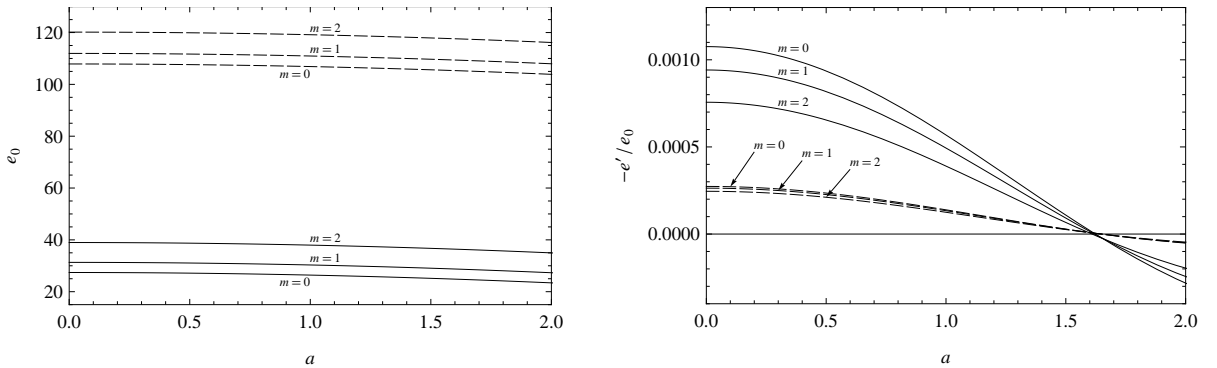


FIG. 2: Dependence of  $e_0$  and  $-e'/e_0$  on  $a$  at  $b = 1$  and  $r_i = 0.5$ .

Fig. 1 shows in dimensionless units the dependence of the unperturbed energy  $e_0$  and the relative corrections  $-e'/e_0$  to the energy on a magnetic field  $b$  for the fixed values  $a = 1$  and  $r_i = 0.5$ . Fig. 2 demonstrates the dependence of  $e_0$  and  $-e'/e_0$  on the spin-orbit interaction strength  $a$  for fixed values  $b = 1$  and  $r_i = 0.5$ . The solid lines represent

TABLE I: Dependence of  $e_0$  and  $-e'/e_0$  on  $r_i$  at  $a = 1$  and  $b = 1$ .

$r_i$	$m = 0$		$m = 1$		$m = 2$	
	$e_0, \quad (-e'/e_0)$					
0	4.48334 ( $5.1620 \times 10^{-3}$ )	26.952 ( $7.6793 \times 10^{-4}$ )	14.677 ( $1.3573 \times 10^{-3}$ )	45.9308 ( $4.3783 \times 10^{-4}$ )	27.3538 ( $6.5366 \times 10^{-4}$ )	67.4988 ( $2.8612 \times 10^{-4}$ )
0.1	7.72294 ( $2.7373 \times 10^{-3}$ )	36.3199 ( $5.5037 \times 10^{-4}$ )	14.9000 ( $1.3272 \times 10^{-3}$ )	47.1566 ( $4.2162 \times 10^{-4}$ )	27.3612 ( $6.5331 \times 10^{-4}$ )	67.5757 ( $2.8550 \times 10^{-4}$ )
0.5	26.4059 ( $5.6869 \times 10^{-4}$ )	106.878 ( $1.3905 \times 10^{-4}$ )	30.3106 ( $4.9374 \times 10^{-4}$ )	110.949 ( $1.3401 \times 10^{-4}$ )	37.9733 ( $3.9012 \times 10^{-4}$ )	119.165 ( $1.2497 \times 10^{-4}$ )
0.9	218.124 ( $3.8120 \times 10^{-5}$ )	404.788 ( $-1.2162 \times 10^{-5}$ )	221.245 ( $3.7573 \times 10^{-5}$ )	407.21 ( $-2.0380 \times 10^{-5}$ )	226.608 ( $3.6655 \times 10^{-5}$ )	410.102 ( $-1.9963 \times 10^{-5}$ )

the first energy levels and the dashed lines represent the second levels for three values of the angular momentum ( $m = 0, 1, 2$ ).

Table 1. illustrates the significant dependence of the unperturbed energy  $e_0$  and the relative correction  $-e'/e_0$  on the relative width  $r_i$  of the quantum ring at the fixed values  $a = 1$  and  $b = 1$ . The values of  $-e'/e_0$  are placed in parentheses. The value  $r_i = 0$  corresponds to a quantum dot. The table lists only first two energy levels. Note that for  $r_i = 0.9$  already the second energy levels lie above the potential well with the dimensionless depth of  $v = 400$  in the considered case. In the absence of a magnetic field, the bound states occur only at  $e_0 < v$ . The presence of a magnetic field leads to appearance of the discrete energy levels exceeding magnitude of a potential well depth.

## V. CONCLUSION

Wave functions and energy levels are obtained for electrons in two-dimensional quantum rings with taking into account the spin-orbit Rashba and Dresselhaus interactions of equal strengths in the presence of an external uniform constant magnetic field in the framework of an adequate model with a confining potential of finite depth. These results may be of interest in the study of spin-dependent effects in semiconductor heterostructures.

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